§1 Graph Transformations

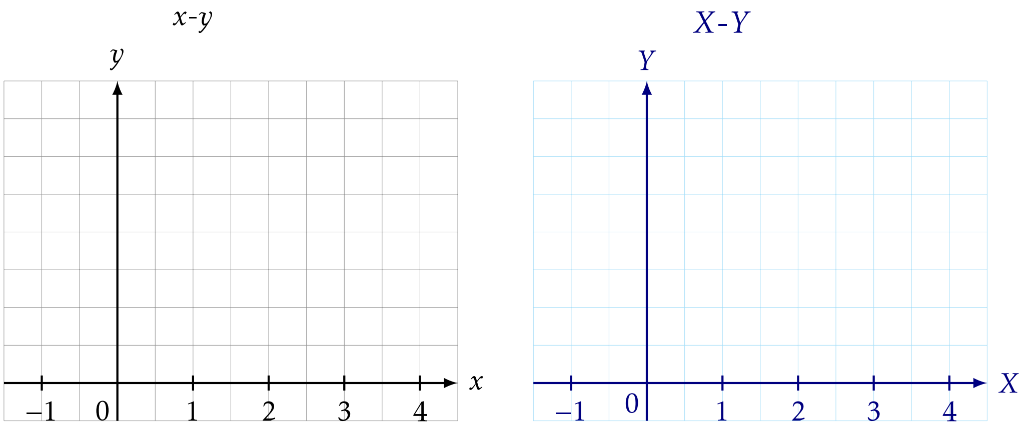
§1.1 Affine Transformations

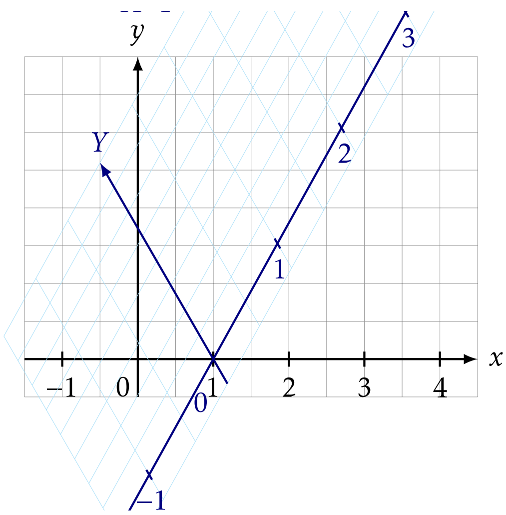
You should already be familiar with performing graph transformations of the form

$$y = f(x) \to y = af(bx+c)+d,$$

including where $a, b < 0$.

We are going to be thinking about these transformations in even more depth, and extend them in several ways. Some you will have seen before, and some might be new to you. The method we will be using involves thinking about multiple co-ordinate grids. The first coordinate grid x-y we might think of as the **world of the page** and the second coordinate grid X-Y we might think of as the **world of the function**.

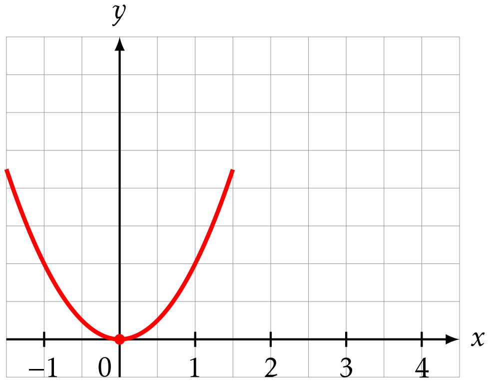




On the following page, we will consider the transformation $y = f(x) \to y = f(x-2)$. The way we will solve this problem is by:

1. Plotting $y = f(x)$ on the **page coordinates**
2. Plotting $Y = f(X)$ on the **function coordinates**
3. Plotting the **page coordinates** on the **function coordinates**
4. Removing the **function coordinates**

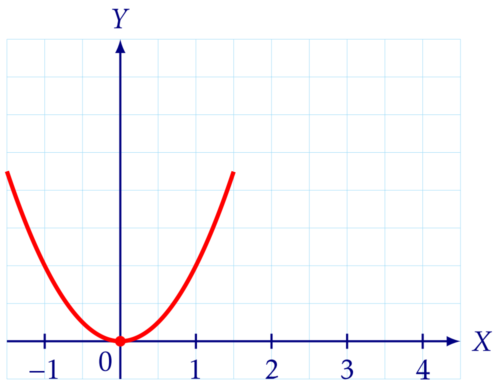
$y = f(x)$ on x-y



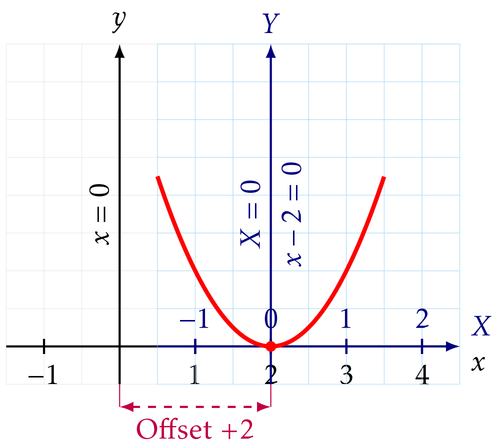
First, we just sketch $y = f(x)$ on our standard co-ordinates. Ideally, we've chosen $f(x)$ so it's easy to sketch, or we're given it by someone else.

Then we sketch $Y = f(X)$ on X-Y coordinates. But this should be exactly the same

$Y = f(Y)$ on X-Y



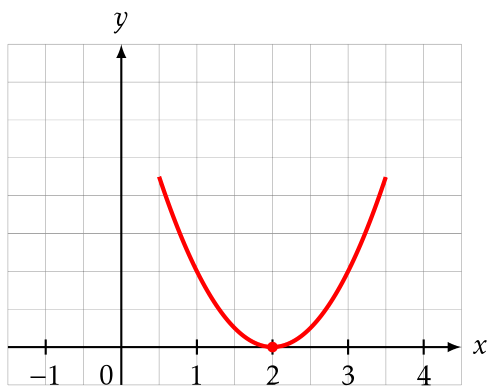
$Y = f(X)$ on X-Y and x-y



Next we overlay the X-Y coordinates on the original x-y grid. Since everything here is 'nice' we don't need to think too hard about how the grid behaves, just focussing on the grid lines. The Y-axis, which has equation $X = 0$ will coincide with the line $X = x - 2 = 0$ or $x = 2$. Something similar will happen for all other lines.

Finally, we remove the X-Y coordinates; we no longer need them. We can see that our function has translated 2 units in the positive x-direction. Hopefully, what we expected.

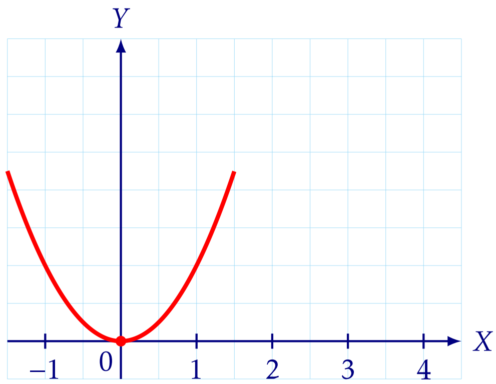
Result: $y = f(x-2)$



This might have all seemed rather convoluted for a transformation which you already knew how to do. However, as we will see in the following pages, it is a framing which is very powerful. (Even if we don't need to do all steps simultaneously).

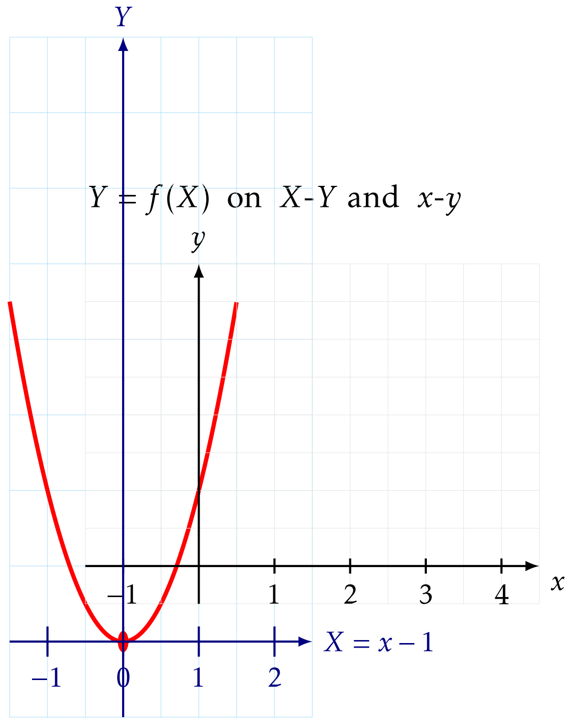
Let's try a more complicated example, $y = f(x) \to y = 2f(x+1) - 1$.

$Y = f(Y)$ on X-Y

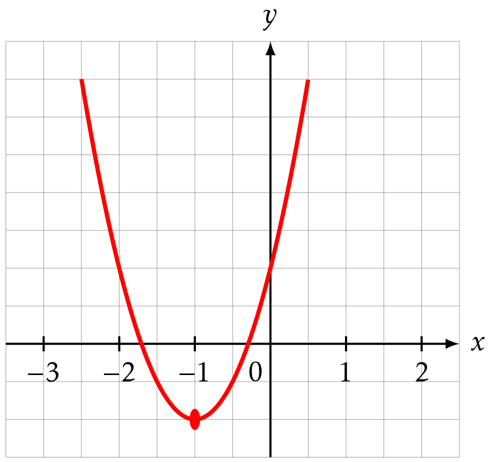


First we rearrange to get $Y = f(X)$ where $Y = \frac{y+1}{2}$ and $X = x + 1$, but we still sketch $Y = f(X)$

Next we overlay the X-Y coordinates on the original x-y grid. We need to think slightly harder to see how our grid transforms. Notice that the Y-axis will be at $X = 0 \Leftrightarrow x = -1$, the line $X = 1$ will be at $x = 0$, so the vertical lines will be unchanged. However, when we consider the X axis which is at $Y = 0$, this will now be at $y = -1$, but the line $Y = 1$ will be at $y = 1$ and $Y = 2$ will be at $y = 3$, so the horizontal grid lines expand apart, representing the vertical stretch



$y = 2f(x+1) - 1$



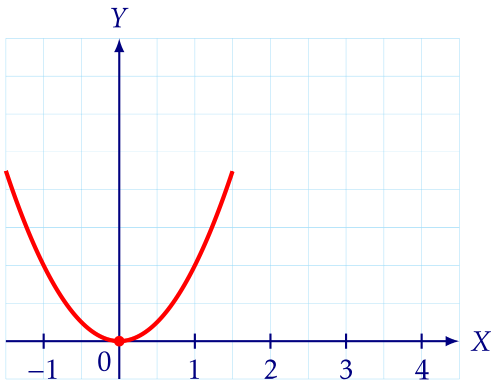
Finally, we remove the X-Y coordinates; we no longer need them.

Example

Sketch $y = (x - y)^2$

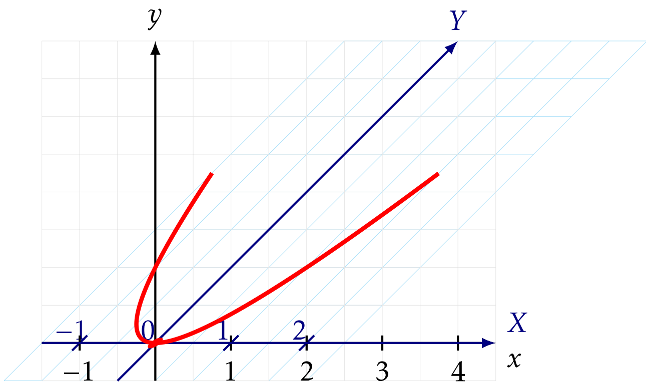
Let's say $f(x) = x^2$ and our equation is $y = f(x - y)$

$Y = f(Y)$ on X-Y



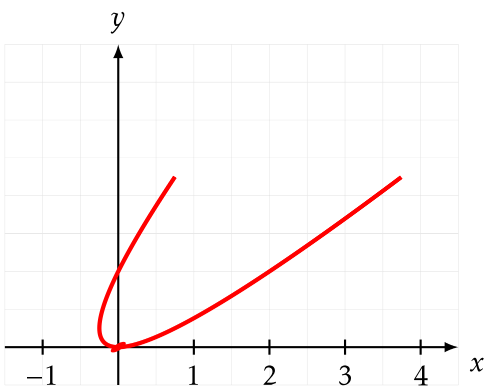
So we have $Y = y$ is $X = x - y$,

$Y = f(X)$ on X-Y and x-y



Next we overlay the X-Y coordinates on the original x-y grid. We need to think slightly harder to see how our grid transforms. Notice the lines $Y = c$ and $y = c$ will be the same, so those lines are unchanged. However $X = c$ becomes the line $x - y = c$. For example $X = 0$ will become $y = x$.

$y = f(x - y)$ on x-y



Finally, we remove the X-Y coordinates; we no longer need them.

This was probably already in your toolbox, (albeit possibly using other methods). However, this method may help you with the following questions:

§1.1.1 Exercises

1. The function $y = f(x)$ has a single peak at $(2,4)$ and intercepts the x-axis at 0 and 4. Without finding an algebraic expression for $f(x)$, sketch the following graphs, clearly marking the new position of the peak and the intercepts:
   a) $y = f(2x)$
   b) $y = f(2-x)$
   c) $y = 3 - f(x)$
   d) $x = f(y)$ (Careful: Is this a function?)

2. a) Sketch the region defined by $|x| + |y| = 1$.
   b) By considering the transformation $X = x + y$ and $Y = x - y$, or otherwise, sketch the region defined by:
   $$|x+y| + |x-y| = 2$$
   c) Sketch the curve given by equation $\sqrt{x+y} + \sqrt{x-y} = 1$. (Hint: Consider the domain carefully).

3. Sketch the graph of $8y = x^3 - 12x$ for $-4 \leqslant x \leqslant 4$, marking the coordinates of the turning points. Similarly marking the turning points, sketch the corresponding graphs in the (X,Y)-plane, if
   $$\begin{array}{lll} \text{(a)} & X = \tfrac{1}{2}x, & Y = y, \\ \text{(b)} & X = x, & Y = \tfrac{1}{2}y, \\ \text{(c)} & X = \tfrac{1}{2}x + 1, & Y = y, \\ \text{(d)} & X = x, & Y = \tfrac{1}{2}y + 1. \end{array}$$
   Find values for a, b, c, d such that, if $X = ax + b$, $Y = cy + d$, then the graph in the (X,Y)-plane corresponding to $8y = x^3 - 12x$ has turning points at $(X,Y) = (0,0)$ and $(X,Y) = (1,1)$.
   STEP 1989 Paper 1 Q9

4. (i) x_2 and y_2 are defined in terms of x_1 and y_1 by the equation
   $$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$
   G_1 is the graph with equation
   $$\frac{x^2}{9} + \frac{y^2}{4} = 1$$
   and G_2 is the graph with equation
   $$\frac{\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{9} + \frac{\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2}{4} = 1$$
   Show that, if (x_1, y_1) is a point on G_1, then (x_2, y_2) is a point on G_2. Show that G_2 is an anti-clockwise rotation of G_1 through $45°$ about the origin.

   (ii) i. The matrix
   $$\begin{pmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{pmatrix}$$
   represents a reflection. Find the line of invariant points of this matrix.

ii. Sketch, on the same axes, the graphs with equations

$$y = 2^x \text{ and } 0.8x + 0.6y = 2^{-0.6x+0.8y}$$

(iii) Sketch, on the same axes, for $0 \leq x \leq 2\pi$, the graphs with equations

$$y = \sin x \text{ and } y = \sin(x - 2y)$$

You should determine the exact co-ordinates of the points on the graph with equation $y = \sin(x - 2y)$ where the tangent is horizontal and those where it is vertical.

STEP 2025 Paper 3 Q4

5. Determine the different kinds of conics represented by the equation

$$x^2 + 4xy + 4y^2 + 2(1 + \lambda)x + 8y + 5 + 2\lambda = 0,$$

for different values of λ. Find any critical values of λ

6. Sketch the curve:

$$(x + y)(x^2 + y^2) = 2xy$$

Find the area of the loop *CCE 1957 Calculus (Kings, ...) Q6*

7. A mapping of the (X, Y) plane onto the (x, y) plane is given by

$$x = \sin X \cosh Y, \quad y = \cos X \sinh Y.$$

Find and sketch curves in the (x, y) plane which correspond under this mapping to the lines

a) $X = c$

b) $Y = c$

Sketch these two families on the same axes. To which curves in the (X, Y) plane to the lines $x = 0, y = 0$ and $x = y$ correspond *CCE 1969 Paper 2 Q14*

§1.2 Modulus Function

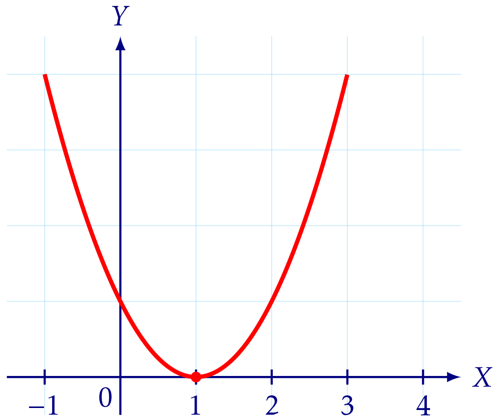
You should already be familiar with the function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Example

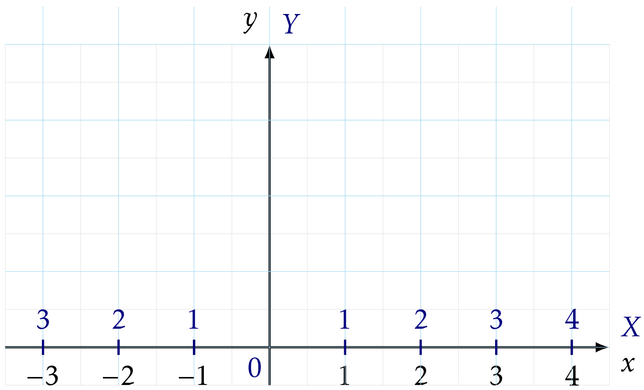
Sketch $y = (|x| - 1)^2$

$Y = f(Y)$ on X-Y



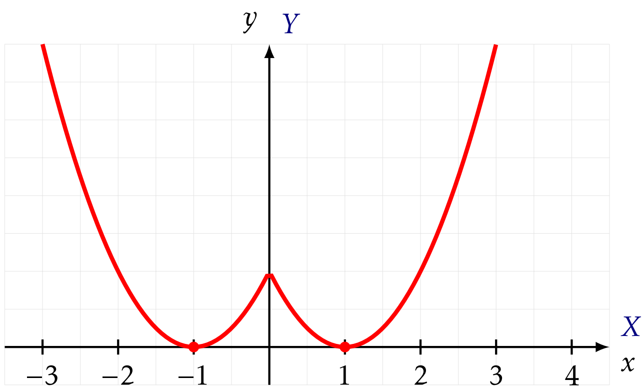
Let $f(x) = (x-1)^2$, (this is simple enough for us to sketch, either by knowing about quadratics or from previous work on transformations). So we have $Y = y$ is $X = |x|$

$Y = f(X)$ on X-Y and x-y



The lines $X = c > 0$ appear *twice* on the x-y coordinate grid, at $x = c$ and $x = -c$. The lines $X = c < 0$ don't appear anywhere.

$Y = f(X)$ on X-Y and x-y



Putting the curve back in from the X-Y grid, we can see it is reflected in the y-axis.

§1.2.1 Exercises

1. Sketch the following functions in the interval $-2 \le x \le 2$:
   a) $y = \exp\left(-(x^2-1)^2\right)$
   b) $y = \exp\left(-|x^2-1|\right)$

   What do you notice about the derivative at the maxima and minima

2. Sketch $y = x^2 - 4|x| + 3$

3. Sketch

   a) $y = \sin|x|$

   b) $|y| = \sin x$

4. Sketch the following subsets of the x-y plane:

   a) $|x| + |y| \le 1$;
   b) $|x-1| + |y-1| \le 1$;
   c) $|x-1| - |y+1| \le 1$;
   d) $|x|\,|y-2| \le 1$.

STEP 1999 Paper 1 Q4

5. (i) The function f is defined by $f(x) = |x-a| + |x-b|$, where $a < b$. Sketch the graph of $f(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $g(x)$, where $g(x) = |2x - a - b|$. What shape is the quadrilateral with vertices $(a,0)$, $(b,0)$, $(b, f(b))$ and $(a, f(a))$?

   (ii) Show graphically that the equation

   $$|x-a| + |x-b| = |x-c|,$$

   where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

   (iii) For the equation

   $$|x-a| + |x-b| = |x-c| + |x-d|,$$

   where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

STEP 2014 Paper 2 Q7

6. The function f is defined by

   $$f(x) = |x-1|,$$

   where the domain is $\mathbb{R}$, the set of all real numbers. The function $g_n = f^n$, with domain $\mathbb{R}$, so for example $g_3(x) = f(f(f(x)))$. In separate diagrams, sketch graphs of g_1, g_2, g_3 and g_4. The function h is defined by

   $$h(x) = \left|\frac{\sin \pi x}{2}\right|,$$

   where the domain is $\mathbb{R}$. Show that if n is even,

   $$\int_0^n \left(h(x) - g_n(x)\right) \mathrm{d}x = \frac{2n}{\pi} - \frac{n}{2}.$$

STEP 2003 Paper 2 Q6

7. What is the area of the polygon formed by all points (x,y) in the plane satisfying the inequality

   $$||x|-2| + ||y|-2| \le 4?$$

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§1.3 Reciprocals

We now consider the transformation $y = \frac{1}{f(x)}$.

In the language of our coordinate grids, we can view this as plotting $Y = f(X)$ on a standard x-grid, but where the Y-axis has been non-linearly distorted.

- Large values of Y become small values of y.
- Values of Y close to 0 shoot off to infinity in y.

While drawing a distorted grid is physically difficult, we can identify specific **invariant lines** and **key features** that act as the skeleton for our new graph.

Key points to consider

When sketching $y = \frac{1}{f(x)}$ based on the graph of $f(x)$:

1. **Roots become Asymptotes:**
$$f(x) \to 0 \implies y \to \pm\infty$$
If $f(x)$ has a root at $x = a$, the reciprocal has a vertical asymptote $x = a$. You must check the sign of $f(x)$ on either side of a to determine if $y \to +\infty$ or $-\infty$.

2. **Asymptotes become Roots:**
$$f(x) \to \pm\infty \implies y \to 0$$
If $f(x)$ grows without bound, the reciprocal approaches the x-axis.

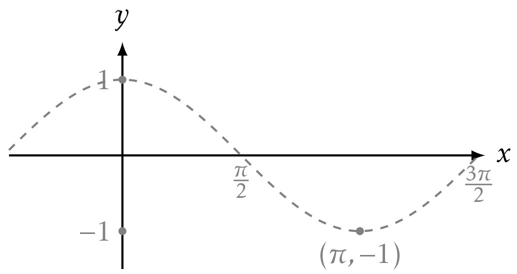
3. **Invariant Points:**
$$f(x) = 1 \implies y = 1 \quad \text{and} \quad f(x) = -1 \implies y = -1$$
The new curve must intersect the old curve whenever the y-value is ± 1.

4. **Turning Points Invert:** Using the chain rule, $\frac{dy}{dx} = -\frac{f'(x)}{(f(x))^2}$.
   - If $f'(x) = 0$, then $y' = 0$ (provided $f(x) \neq 0$).
   - A local **maximum** on $f(x)$ becomes a local **minimum** on $1/f(x)$ (and vice versa) [assuming of course we are not talking about turning points on the x-axis]

Example

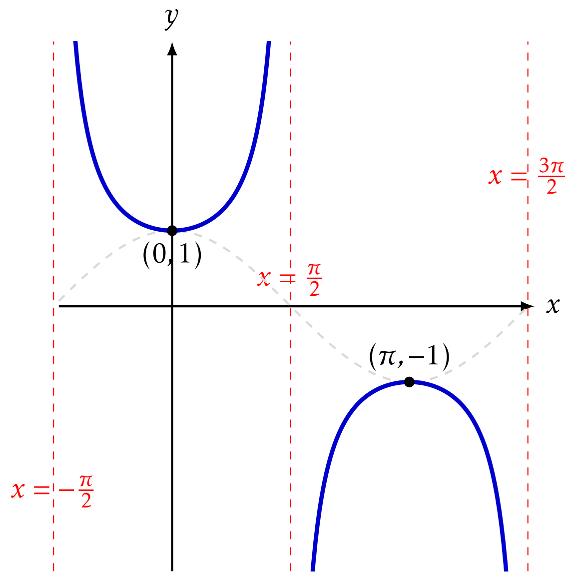
Sketch $y = \sec x$ by considering $f(x) = \cos x$.

$y = f(x)$



We start by sketching $y = f(x) = \cos x$
We identify the roots at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.

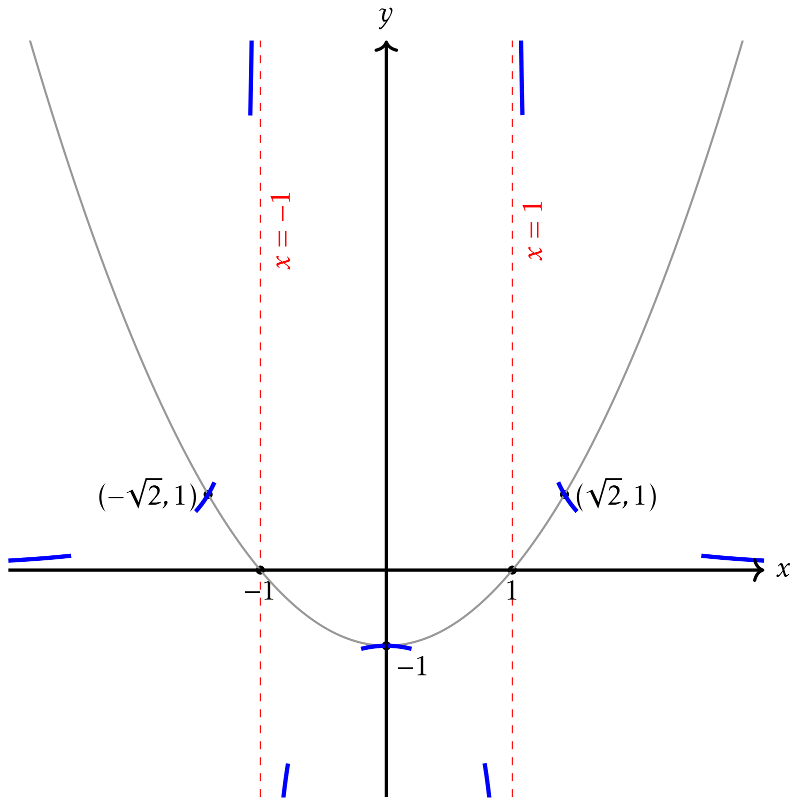
1. **Roots→ asymptotes**. We draw asymptotes at $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
2. **Asymptotes→ roots**. $\cos x$ has no asymptotes
3. **Invariant points**. (Also our turning points)
4. **Maxima ↔ minima**: we must have a ∪ and ∩ shapes around $-\pi, 0, \pi, \ldots$



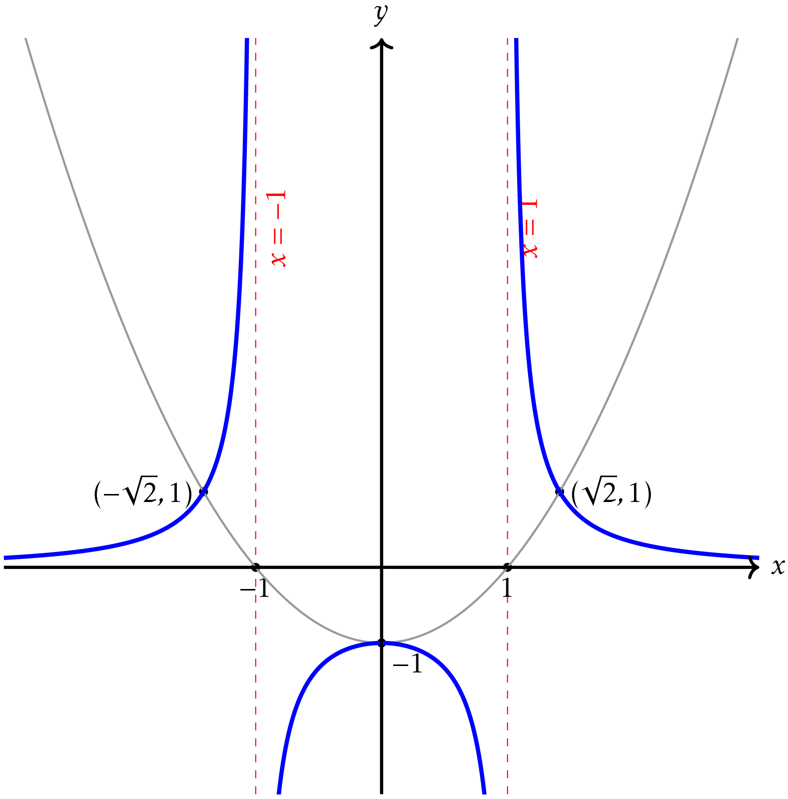
Example
Sketch $y = \frac{1}{x^2-1}$

First we sketch $y = x^2 - 1$

1. **Roots → Asymptotes**: The graph of the reciprocal has vertical asymptotes at $x = 1$ and $x = -1$.
2. **Asymptotes → Roots**: As $x \to \infty$, $x^2 - 1 \to \infty$, so $y \to 0$ from the positive side.
3. **Invariant Points**: $x^2 - 1 = -1 \Rightarrow x = 0$. $x^2 - 1 = 1 \Rightarrow x = \pm\sqrt{2}$ The graphs intersect at $(0, -1), (\pm\sqrt{2}, 1)$
4. **Maxima ↔ minima**, the point $(0, -1)$ will be a map to a maximum



Once we have made these guiding marks, and since we know there aren't any other turning points to worry about, we can simply join everything together with a smooth curve. With practice, this becomes a very fast process.


y
x
x = −1
x = 1
(−√2, 1)
(√2, 1)
−1
1
−1

§1.3.1 Exercises

1. Sketch the following curves on separate axes, clearly indicating the position of any turning points and asymptotes.

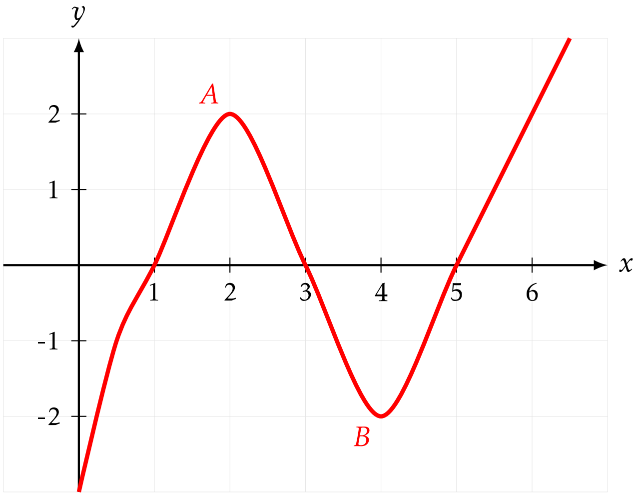
   a) $y = \operatorname{cosec}(x)$ for $0 < x < 4\pi$

   b) $y = \cot(x)$ for $0 < x < 2\pi$

   c) $y = \frac{1}{x^2+1}$

   d) $y = \frac{1}{x^2-3x+2}$

2. The graph of $y = f(x)$ is shown below. Sketch $y = \frac{1}{f(x)}$.



3. a) Sketch $y = x + \frac{1}{x}$.

   b) Hence or otherwise, sketch $y = \frac{x}{x^2+1}$.

4. a) By performing a polynomial division, show that $\frac{2x+3}{x-1}$ can be written in the form $A + \frac{B}{x-1}$.

   b) Hence, using translations and reciprocal transformations, sketch $y = \frac{2x+3}{x-1}$.

5. (i) Sketch the graph of $y = \frac{1}{x}$.

   (ii) Sketch the graph of $y = \frac{1}{x-a} + \frac{1}{x-b}$ where $b > a > 0$. Determine the coordinates of the point where the gradient is zero.

   (iii) Show that the equation

$$\frac{1}{x-a} + \frac{1}{x-b} = c$$

   has two real roots for all $c \neq 0$.

6. Sketch the curve with equation

$$y = \frac{1}{|x-1| - |x-3|}$$

   Carefully marking any vertical or horizontal asymptotes.

7. If $x > 0$, prove that $(x-1)^2$ is not less than $x(\log x)^2$.

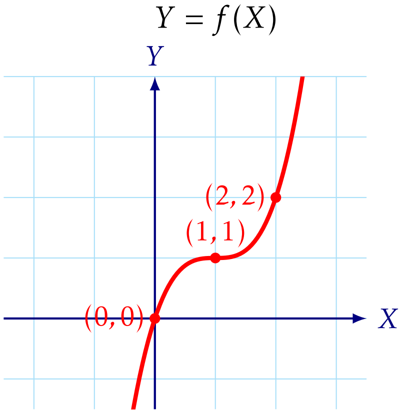
   Discuss the general behaviour of the function $(\log x)^{-1} - (x-1)^{-1}$ for positive values of x and with special reference to $x = 1$.

   Sketch the graph of the function. *CCE 1940 (St Catz, ...) Algebra and Calculus Q5*

§1.4 Non-linear Substitutions

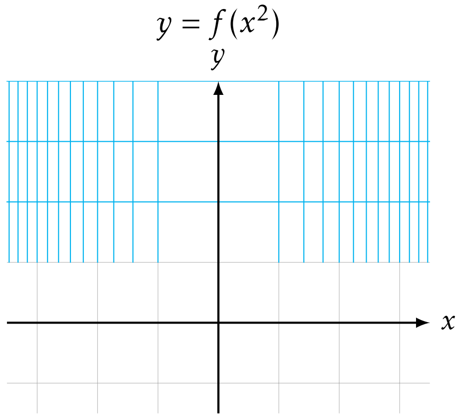
Example

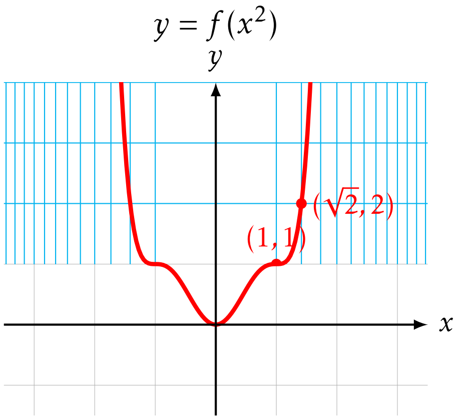
Given the graph of $y = (x-1)^3 + 1$, sketch $y = (x^2-1)^3 + 1$.



As usual, we sketch $Y = f(X)$, wth $Y = y$ and $X = x^2$

The line $X = c$ is the same as $x^2 = c$, ie the lines $x = \pm\sqrt{c}$, so we can see in the range -4 to 4 we end up with the lines $X = 0, 1, 2, \ldots 16$.
I have just drawn the grids on the top half and bottom half to illustrate this phenomenon.





So the graph has been reflected in the y-axis, and 'stretched' between -1 and 1 and 'squashed' elsewhere. Note that the squashing and stretching are non-linear!

We don't necessarily need to go through this whole process every time we sketch $y = f(x^2)$, as long as we think about some simple rules:

1. Any value of $x < 0$ becomes irrelevant
2. Our final graph will be symmetric in $x = 0$ (the y-axis)
3. We apply a non-linear 'squash' (which expands $(-1, 1)$ and compresses everything else towards the y-axis)

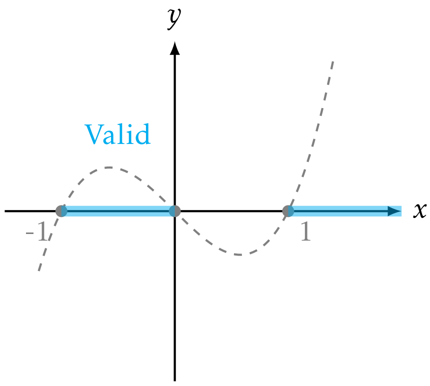
4. $\frac{dy}{dx} = f'(x^2) \cdot 2x$ so stationary points map to stationary points, and we should expect the origin to be a stationary point (unless $f'(0)$ doesn't exist)

Example

Sketch $y^2 = x^3 - x$.

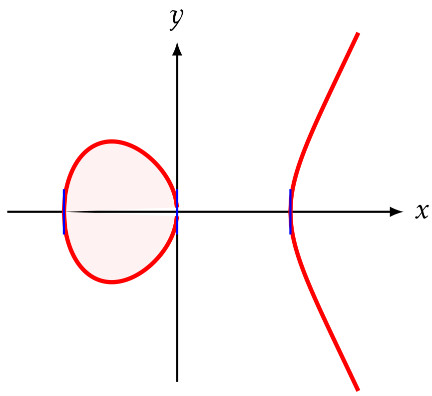
The first thing we should realise by now, is that $X = x^2, Y = y$ and $X = x, Y = y^2$ are two sides of the same coin, and so we should be able to approach problems of the form $y^2 = f(x)$ confidently. However, there's one subtlety which is worth noting, so pay attention to the gradient at roots.

$y = f(x)$



First, sketch the curve $y = x^3 - x$.
We are only interested in where $y \geq 0$, so the curve exists for $-1 \leq x \leq 0$ and $x \geq 1$.

$y^2 = x^3 - x$

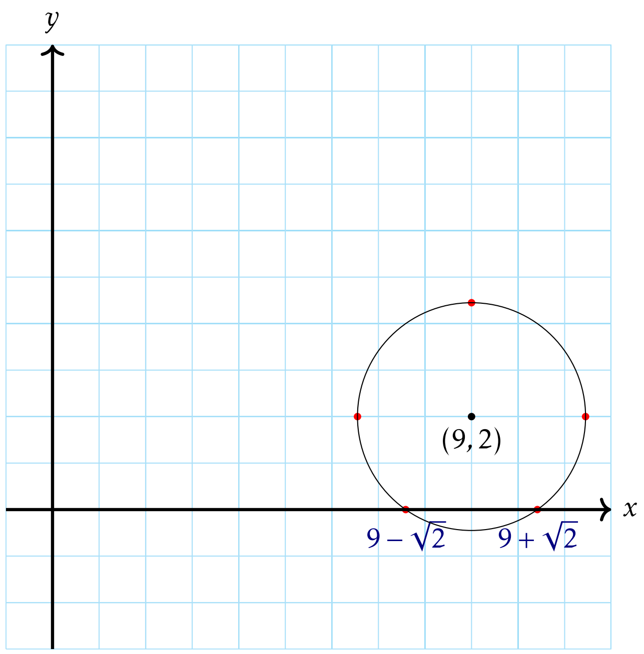


Now we reflect our curve in the x-axis, as well as applying a stretch.
Notice that if $y^2 = f(x)$ then $2y\frac{dy}{dx} = f'(x) \Rightarrow \frac{dy}{dx} = \frac{1}{2y}f'(x)$, therefore assuming $f'(x) \neq 0$, when $y = 0$ the slope is vertical.
Finally, notice that maxima and minima map to themselves (for the same algebraic reasoning as above).
This shape is a classic **elliptic curve**.

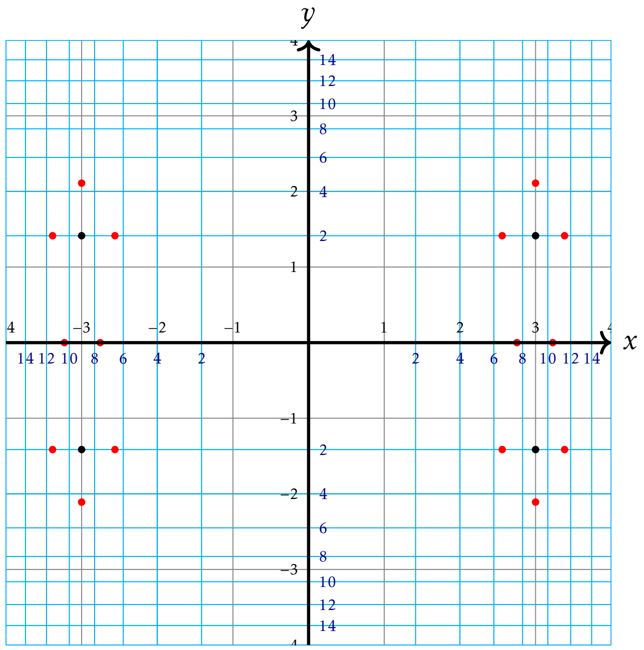
Example (CCE 1958 Paper 4 Q6)

Sketch the curve $(x^2-9)^2+(y^2-2)^2=6$.

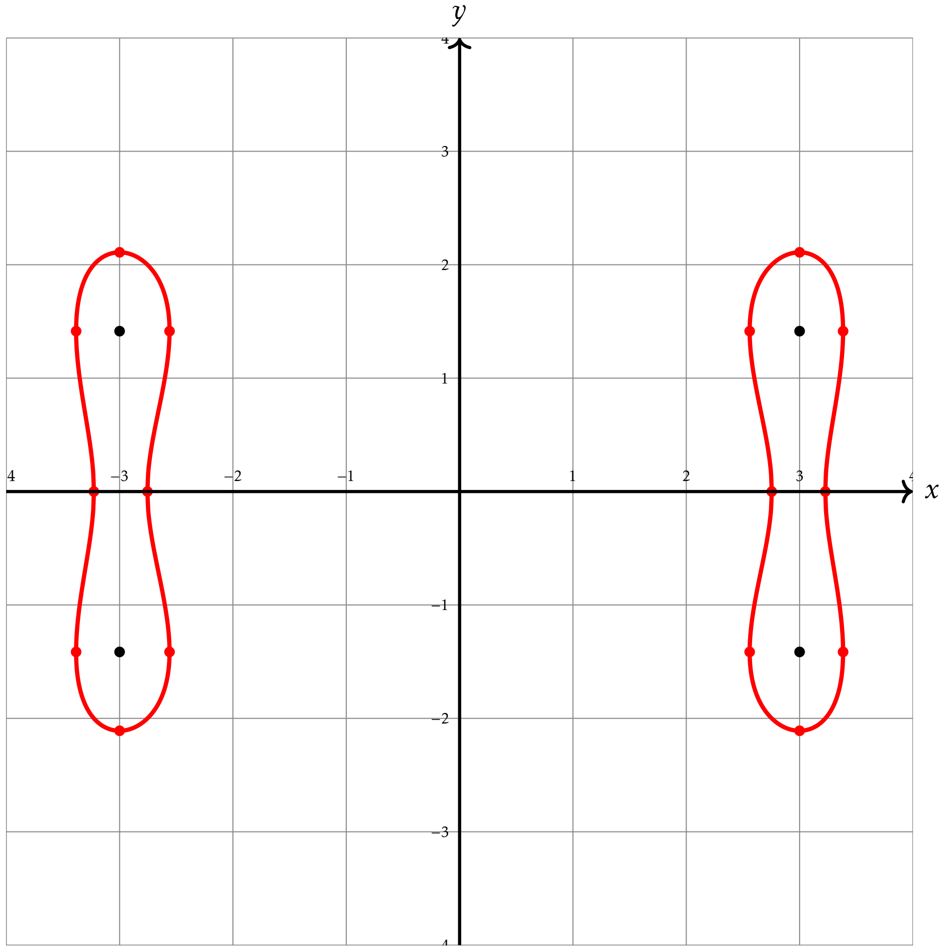
It should be clear by now what transformation we are going to use: $X=x^2, Y=y^2$, so first we will sketch $(X-9)^2+(Y-2)^2=6$



Notice our grid is going to get transformed in *both* directions, so we will just draw our grid first as well as adding a few key points (turning points, intersection of axes, and the centre of the circle).



$$(x^2 - 9)^2 + (y^2 - 2)^2 = 6$$

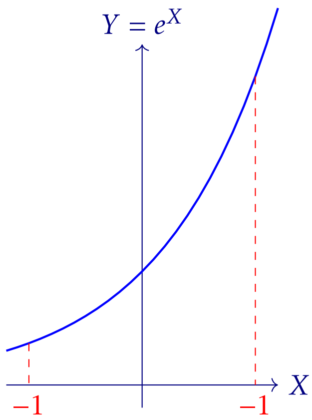


Notice how the lines are smooth (and vertical) as the cross the x-axis.

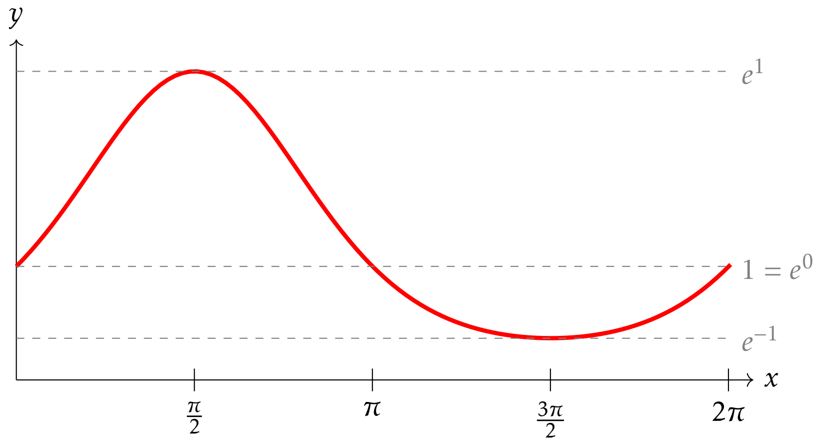
Example

Sketch $y = e^{\sin x}$.

When considering sketches of the form $y = f(\sin x)$ or $y = f(\cos x)$, it's worth noting that regardless of the value of x inputted, the value of X will be between -1 and 1. Secondly, we will end up with some output which is periodic in x.



The Source
$[-1, 1]$



We start by noting on $[0, \frac{\pi}{2}$ we must take the values of $f(0)$ to $f(1)$. On $[\frac{\pi}{2}, \pi]$ we must take those same values *with symmetry*, before on $[\pi, \frac{3\pi}{2}]$ taking the values of $f(0)$ to $f(-1)$, before repeating those again.

§1.4.1 Exercises

1. Sketch the following curves, carefully marking the domain and range:

   a) $y = \ln(1 - x^2)$

   b) $y = \sqrt{\sin x}$

   c) $y = \sin(x^2)$

   d) $y^2 = \sin x$

2. Sketch the curve $y^2 = x^2(1 - x^2)$.

3. a) Sketch $y = x^3 - 3x + 2$.

   b) Hence sketch $y^2 = x^3 - 3x + 2$.

   c) Describe the behaviour of the curve $y^2 = x^3 - 3x + 2$ at the point $(1, 0)$. Is there a vertical tangent?

4. Sketch the graph of $y = f(x)$ where $f(x) = \frac{1}{\sqrt{x^4 - 1}}$.

5. (i) Sketch the graph of $y = \cos(\sin x)$.

   (ii) Sketch the graph of $y = \sin(\cos x)$.

   (iii) Which of these two functions has the larger amplitude?

6. Sketch the curve $y^2 = x^3(1 - x^2)$. From your sketch, estimate the number of times the line $y = ax$ cuts the curve for various values of the constant a. Find the range of values of a for which the line $y = ax$ cuts the curve in exactly one point other than the origin. *CCE 1975 Paper 3 Q8*

7. Sketch the curve $x^2 = (y - k)^2(y - 2k)$, where x, y are real variables and k is constant, in the three cases (i) $k < 0$, (ii) $k = 0$, (iii) $k > 0$. Describe the nature of the singularity in each case. *CCE 1960 Paper 1 Q1*

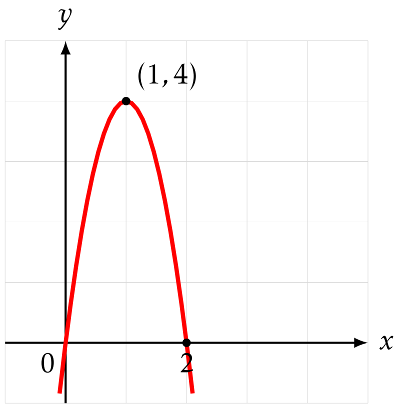
8. Sketch the curve

$$(y^2 - 1)^2 - x^2(2x + 3) = 0.$$

CCE 1955 Paper 2 Q10

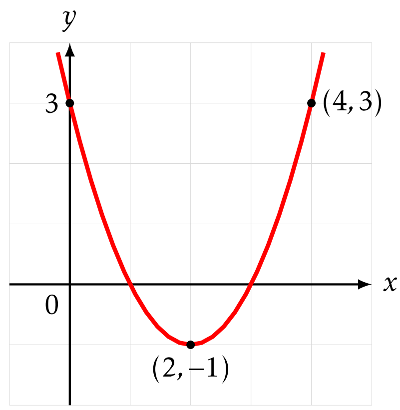
§1.4.2 Answers: Affine Transformations

1. a)



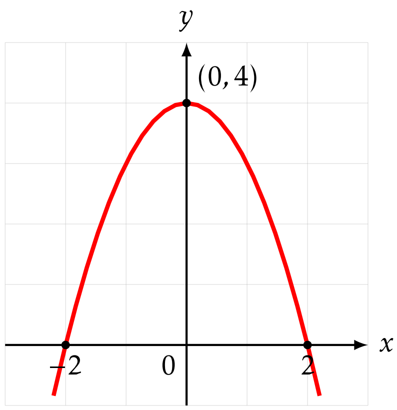
$y = f(2x)$

c)



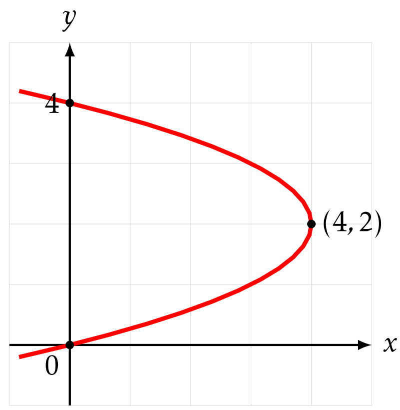
$y = 3 - f(x)$

b)



$y = f(2 - x)$

d)

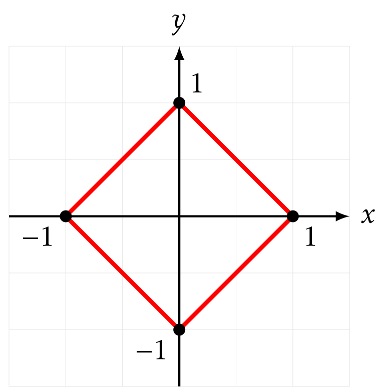


$x = f(y)$

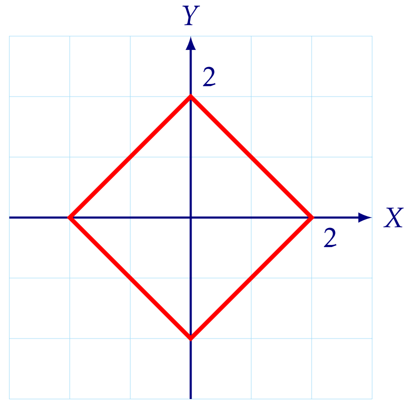
2. a) This is a standard curve known as the ℓ_1 norm unit circle. We can sketch it by considering the four quadrants (cases) to remove the modulus signs.

- **Q1** $(x > 0, y > 0)$: $x + y = 1$
- **Q2** $(x < 0, y > 0)$: $-x + y = 1$
- **Q3** $(x < 0, y < 0)$: $-x - y = 1$
- **Q4** $(x > 0, y < 0)$: $x - y = 1$

These are four line segments connecting the intercepts $(1,0), (0,1), (-1,0), (0,-1)$.



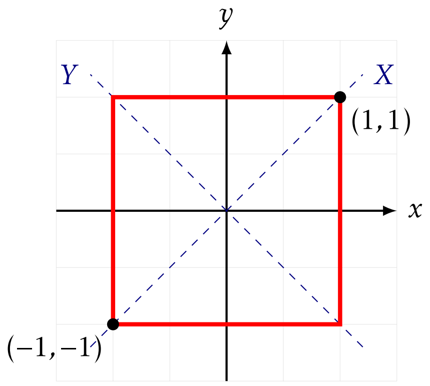
b) We use the coordinate transformation suggested: $X = x + y$ and $Y = x - y$. The equation becomes $|X| + |Y| = 2$.



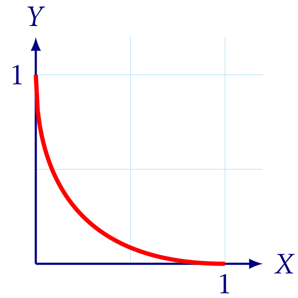
We map the X-Y axes onto the x-y coordinates.

- The X-axis ($Y = 0$) is the line $x - y = 0 \implies y = x$.
- The Y-axis ($X = 0$) is the line $x + y = 0 \implies y = -x$.

Our coordinate system is rotated by 45°.



c) Again, let $X = x + y$ and $Y = x - y$. The equation becomes $\sqrt{X} + \sqrt{Y} = 1$.



The *positive* X-axis, $Y = 0, X \geq 0$ is $x - y = 0, x + y \geq 0 \Rightarrow y = x \geq 0$
The *positive* Y-axis, $X = 0, Y \geq 0$ is $x + y = 0, x - y \geq 0 \Rightarrow x = -y \geq 0$